JOURNAL OF APPROXIMATION THEORY 80, 321–351 (1995)

Norm Estimates for Interpolation Methods Defined by Means of Polygons

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Communicated by Paul L. Butzer

Received September 22, 1992; accepted in revised form December 19, 1993

We study interpolation methods associated to polygons and establish estimates for the norms of interpolated operators. Our results explain the geometrical base of estimates in the literature. Applications to interpolation of weighted L_p-spaces are also given.

INTRODUCTION

This paper deals with interpolation spaces, a theory that since its origin in the early sixties has had a deep interplay with approximation theory as can be seen in the article by Peetre [15] and in the books by Butzer and Berens [3], Bergh and Löfström [1], Triebel [17], and Brudnyĭ and Krugljak [2].

* Supported in part by U.S.–Spain Joint Committee for Cultural and Educational Cooperation (Grant II-C 91024) and DGICYT 90-187.

† Supported in part by DGICYT 90-187.

0021-9045/95 $6.00

The classical real interpolation method, mainly in the form of a K-space, is particularly useful in the context of approximation theory.

We shall work here with interpolation methods similar to the classical real method, but defined for N-tuples ($N \geq 3$) of Banach spaces intead of couples and incorporating some geometrical elements which are essential in developing their theory. These methods were introduced recently by Peetre and one of the present authors in [7].

Previous investigations on interpolation methods for N-tuples have appeared all through the development of interpolation theory. We refer, for example, to the papers written by Foiaş and Lions [12], Yoshikawa [18], Favini [9], Sparr [16], Fernandez [10, 11], Cwikel and Janson [8], Cobos and Peetre [6], and the monograph by Brudnyĭ and Krugljak [2].

Several basic results of classical methods for couples are no longer true in this multidimensional framework. Perhaps the most notorious one is that the equivalence between the K- and J-constructions fails. However, interpolation methods for N-tuples still have important application in analysis. For instance, multidimensional methods are very useful in investigating function spaces with dominating mixed derivatives (see [16, 2]). Such function spaces were introduced by S. M. Nikol'skij around 1963. Contributions to their theory are also due to Lizorkin, Džabrailov, Grisvard, and Besov among other authors (see [16, 17] for complete references).

The interpolation methods we consider here are defined by means of a convex polygon $\Pi = \overline{P_1 \cdots P_N}$, an interior point (α, β) of Π and two scalar parameters t, s. The Banach spaces $A_1, ..., A_N$, which compose the N-tuple to be interpolated, should be thought of as sitting on the vertices of Π.

Although these methods were introduced in [7], the idea of developing such investigation was suggested by Peetre at the Conference on Interpolation Spaces held at Lund in 1982. The geometrical approach that we follow closes the gap between the ideas of real and complex interpolation. It also gives a unified point of view for the multidimensional methods of the type of the classical real method. In particular, when the polygon Π is equal to the simplex, these methods give back (the first nontrivial case of) spaces studied by Sparr [16], and if Π coincides with the unit square we recover those considered by Fernandez [10]. The resulting theory for methods associated to polygons highlights the geometrical aspects of the classical theory of real interpolation (see, for example, [7, 4]).

Our target in Section 1 is to establish estimates for the norms of interpolated operators. This is achieved by minimizing the function

$$\varphi(t, s) = \max_{1 \leq j \leq N} \{ t^{x_j - \alpha} s^{y_j - \beta} M_j \},$$

where (x_j, y_j) are the coordinates of the vertex P_j and M_j stands for the norm of the restriction of the operator to A_j. The outcome is an estimate for the norm of the interpolated operator by a maximum of products of powers of the form $M_i^{c_i} M_j^{c_j} M_k^{c_k}$, where (c_i, c_j, c_k) are the barycentric coordinates of the point (α, β) with respect to the vertices P_i, P_j, P_k. Therefore the classical convexity inequality $M \leqslant M_0^{1-\theta} M_1^{\theta}$ valid for couples, turns now into something of the kind of Caratheodory's theorem on convex sets. Our result describes the geometrical base of norm inequalities already known for Sparr and for Fernandez spaces.

Ideas used to derive the norm estimates are also useful to interpolate L_p-spaces with weights. This is worked out in Section 2.

The problem is now to compute the K- and the J-functional (associated to the polygon Π) for weighted L_p-tuples. Thus we are dealing with a kind of minimum and of maximum problem, respectively.

Among other results we show new differences between the theory of Fernandez spaces and the theory of Sparr spaces. We give a 4-tuple of scalar valued weighted l_1-spaces where the J- and K-methods do not agree. It is known that Sparr's J- and K-methods always coincide in such a situation. We also apply our results on interpolation of weighted L_p-spaces to discuss the influence of the polygon on the resulting interpolation spaces.

In the final Section 3 we come back to norm inequalities but now we are interested in operators acting from a J-space into a K-space. We derive an estimate that involves all norms M_j raised to powers θ_j, where $\bar{\theta} = (\theta_1, ..., \theta_N)$ is any barycentric coordinates of the point (α, β) with respect to (all) vertices of Π.

To do this we relate our polygonal methods with $N-1$ parameters Sparr methods. We prove that if the K- and J-method coincide on an N-tuple then they agree with spaces obtained by using Sparr constructions with $N-1$ parameters and $\bar{\theta}$. In other words, we show that on these N-tuples the theory of methods associated to polygons (in particular Fernandez' theory) is a special case of Sparr's theory. We also give some other applications of this result.

1. Norm Estimates

We begin by recalling definitions of J- and K-spaces associated to polygons (see [7]).

Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon in the plane $\mathbb{R}^2$. The vertices of Π are $P_j = (x_j, y_j)$ $(j = 1, ..., N)$. Let $\bar{A} = \{A_1, ..., A_N\}$ be a Banach N-tuple, that is to say, a family of N Banach spaces all of them continuously embedded in a common linear Hausdorff space. We imagine each space A_j as sitting on the vertex P_j.

By means of the polygon Π we define the following family of norms on $\Sigma(\bar{A}) = A_1 + \cdots + A_N$

$$K(t, s; a) = \inf\left\{ \sum_{j=1}^{N} t^{x_j} s^{y_j} \|a_j\|_{A_j} : a = \sum_{j=1}^{N} a_j,\ a_j \in A_j \right\}.$$

Here t and s stand for positive numbers. Similarly in $\Delta(\bar{A}) = A_1 \cap \cdots \cap A_N$ we can consider the family of norms

$$J(t, s; a) = \max_{1 \leq j \leq N} \{ t^{x_j} s^{y_j} \|a\|_{A_j} \}.$$

Let now (α, β) be an interior point of Π $[(\alpha, \beta) \in \operatorname{Int} \Pi]$ and let $1 \leq q \leq \infty$. The K-space $\bar{A}_{(\alpha, \beta), q; K}$ is defined as the collection of all $a \in \Sigma(\bar{A})$ for which the norm

$$\|a\|_{(\alpha, \beta), q; K} = \left(\int_0^\infty \int_0^\infty (t^{-\alpha} s^{-\beta} K(t, s; a))^q \frac{dt}{t} \frac{ds}{s} \right)^{1/q}$$

is finite.

The J-space $\bar{A}_{(\alpha, \beta), q; J}$ is formed by all those elements $a \in \Sigma(\bar{A})$ for which there exists a strongly measurable function $u = u(t, s)$ with values in $\Delta(\bar{A})$ such that

$$a = \int_0^\infty \int_0^\infty u(t, s) \frac{dt}{t} \frac{ds}{s} \tag{1}$$

and

$$\left(\int_0^\infty \int_0^\infty (t^{-\alpha} s^{-\beta} J(t, s; u(t, s)))^q \frac{dt}{t} \frac{ds}{s} \right)^{1/q} < \infty. \tag{2}$$

The norm on $\bar{A}_{(\alpha, \beta), q; J}$ is

$$\|a\|_{(\alpha, \beta), q; J} = \inf\left\{ \left(\int_0^\infty \int_0^\infty (t^{-\alpha} s^{-\beta} J(t, s; u(t, s)))^q \frac{dt}{t} \frac{ds}{s} \right)^{1/q} \right\},$$

where the infimum is taken over all representations u satisfying (1) and (2).

Let us see some examples:

EXAMPLE 1.1. Assume that Π is equal to the simplex $\{(0,0), (1,0), (0,1)\}$ and that $\alpha > 0$, $\beta > 0$ with $\alpha + \beta < 1$. In this case the K- and the J-functional are

$$K(t, s; a) = \inf\left\{ \|a_1\|_{A_1} + t\, \|a_2\|_{A_2} + s\, \|a_3\|_{A_3} : a = \sum_{j=1}^{3} a_j,\ a_j \in A_j \right\}$$

$$J(t, s; a) = \max\{ \|a\|_{A_1}, t\, \|a\|_{A_2}, s\, \|a\|_{A_3} \}.$$

Spaces defined by means of the simplex and the interior point (α, β) coincide with those studied by Sparr in [16]. Let us denote them by $\bar{A}^S_{(\alpha,\beta),q;K}$ and $\bar{A}^S_{(\alpha,\beta),q;J}$.

EXAMPLE 1.2. Take Π equal to the unit square $\{(0,0), (1,0), (0,1), (1,1)\}$ and $0<\alpha, \beta<1$. The functionals are now

$$K(t, s; a) = \inf\left\{ \|a_1\|_{A_1} + t\,\|a_2\|_{A_2} + s\,\|a_3\|_{A_3} + ts\,\|a_4\|_{A_4} : a = \sum_{j=1}^{4} a_j,\, a_j \in A_j \right\}$$

$$J(t, s; a) = \max\{\|a\|_{A_1}, t\,\|a\|_{A_2}, s\,\|a\|_{A_3}, ts\,\|a\|_{A_4}\},$$

and spaces generated in this way are the same as those considered by Fernandez in [10, 11]. For later use we call them $\bar{A}^F_{(\alpha,\beta),q;K}$ and $\bar{A}^F_{(\alpha,\beta),q;J}$.

The following result shows a sufficient condition for the coincidence of interpolation methods associated to different polygons. The proof is just a change of variables (see [5, Remark 4.1]).

LEMMA 1.3. *Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon, let $(\alpha, \beta) \in \operatorname{Int} \Pi$, $1 \leqslant q \leqslant \infty$ and let R be the mapping defined by*

$$R(u, v) = Q + U(u, v), \qquad (u, v) \in \mathbb{R}^2$$

where $Q \in \mathbb{R}^2$ and U is any linear isomorphism of $\mathbb{R}^2$. Then the K- and the J-spaces defined by means of Π and (α, β) coincide (with equivalent norms) with those defined by means of $R(\Pi) = \overline{RP_1 \cdots RP_N}$ and $R(\alpha, \beta)$.

As we shall see in Section 2 (Example 2.4) if the polygons are not related by any affine isomorphism then the resulting interpolation methods may be different.

Let $\bar{B} = \{B_1, ..., B_N\}$ be another Banach N-tuple which we also think of as sitting on the vertices of another copy of the polygon Π. By $T\colon \bar{A} \to \bar{B}$ we denote a linear operator from $\Sigma(\bar{A})$ into $\Sigma(\bar{B})$ whose restriction to each A_j defines a bounded operator from A_j into B_j ($j = 1, ..., N$).

One can easily check that if $T\colon \bar{A} \to \bar{B}$ then the restriction of T to $\bar{A}_{(\alpha,\beta),q;K}$ gives a bounded linear operator

$$T\colon \bar{A}_{(\alpha,\beta),q;K} \to \bar{B}_{(\alpha,\beta),q;K}. \tag{3}$$

For J-spaces, we have that

$$T\colon \bar{A}_{(\alpha,\beta),q;J} \to \bar{B}_{(\alpha,\beta),q;J} \tag{3'}$$

is bounded too.

Our target is to estimate the norms of (3) and (3'). Write M_j for $\|T\|_{A_j, B_j}$ ($j=1, ..., N$). In the case of the classical real method for Banach couples $(A_0, A_1)_{\theta, q}$, where the J- and K-constructions always agree, the well-known estimate

$$\|T\|_{\bar{A}_{\theta, q}, \bar{B}_{\theta, q}} \leqslant M_0^{1-\theta} M_1^{\theta}$$

is obtained by a simple change of variables in the integrals defining the norm (see [1 or 17]). In our multidimensional case the situation is not so easy because products of powers of parameters t and s might appear in the K- and the J-functionals. However, since

$$K(t, s; Ta) \leqslant \max_{1 \leqslant j \leqslant N} \{\lambda^{x_j} \mu^{y_j} M_j\} \; K(t/\lambda, s/\mu; a)$$

changing variables we get

$$\|Ta\|_{(\alpha, \beta), q; K} \leqslant \max_{1 \leqslant j \leqslant N} \{\lambda^{x_j - \alpha} \mu^{y_j - \beta} M_j\} \; \|a\|_{(\alpha, \beta), q; K}.$$

Hence

$$\|T\|_{\bar{A}_{(\alpha, \beta), q; K}, \bar{B}_{(\alpha, \beta), q; K}} \leqslant \inf_{t>0, s>0} \left[\max_{1 \leqslant j \leqslant N} \{t^{x_j - \alpha} s^{y_j - \beta} M_j\} \right]$$

and the same inequality holds for J-spaces.

Following the notation of [7] we put

DEFINITION 1.4. Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon with $P_j = (x_j, y_j)$, and let $(\alpha, \beta) \in \operatorname{Int} \Pi$. Then for any N non-negative real numbers $M_1, ..., M_N$ we write

$$D_{\alpha, \beta}(M_1, ..., M_N) = \inf_{t>0, s>0} \left[\max_{1 \leqslant j \leqslant N} \{t^{x_j - \alpha} s^{y_j - \beta} M_j\} \right].$$

If we could calculate $D_{\alpha, \beta}(M_1, ..., M_N)$ then we will have an estimate for the norms of interpolated operators (3) and (3'). Let us show some examples:

EXAMPLE 1.5. Let Π be the simplex and let $\alpha > 0$, $\beta > 0$ with $\alpha + \beta < 1$. A direct computation yields that

$$\begin{aligned} D_{\alpha, \beta}(M_1, M_2, M_3) &= \inf_{t>0, s>0} \left[\max\{t^{-\alpha} s^{-\beta} M_1, t^{1-\alpha} s^{-\beta} M_2, t^{-\alpha} s^{1-\beta} M_3\} \right] \\ &= M_1^{1-\alpha-\beta} M_2^{\alpha} M_3^{\beta} \end{aligned}$$

which coincides with Sparr's estimate (see [16]).

EXAMPLE 1.6. Take now Π equal to the unit square and $0<\alpha, \beta<1$. Put

$$N_1 = M_2^{1-\beta} M_3^{1-\alpha} M_4^{\alpha+\beta-1},$$
$$N_2 = M_1^{1-\beta} M_3^{\beta-\alpha} M_4^{\alpha},$$
$$N_3 = M_1^{1-\alpha} M_2^{\alpha-\beta} M_4^{\beta},$$
$$N_4 = M_1^{1-\alpha-\beta} M_2^{\alpha} M_3^{\beta},$$

and define $\mathscr{C}^*$ as the set of those N_j having only non-negative exponents. Then

$$\begin{aligned} D_{\alpha,\beta}(M_1, M_2, M_3, M_4) &= \inf_{t>0,\, s>0} [\max\{t^{-\alpha}s^{-\beta}M_1, t^{1-\alpha}s^{-\beta}M_2, t^{-\alpha}s^{1-\beta}M_3, t^{1-\alpha}s^{1-\beta}M_4\}] \\ &= \max\{N_j : N_j \in \mathscr{C}^*\}. \end{aligned}$$

This result is due to Cobos and Peetre [7, Thm. 2.2], but the proof given there is not complete. One should change "sums" into "maximums" in the definition of the function f in [7, p. 380]. More precisely

$$\begin{aligned} D_{\alpha,\beta}(M_1, M_2, M_3, M_4) &= \inf_{s>0} [\inf_{t>0} \{\max[\max(M_1, sM_3)\, t^{-\alpha}, \max(M_2, sM_4)\, t^{1-\alpha}]\}\, s^{-\beta}] \\ &= \inf_{s>0} [(\max(M_1, sM_3))^{1-\alpha} (\max(M_2, sM_4))^{\alpha} s^{-\beta}] \\ &= M_3^{1-\alpha} M_4^{\alpha} \inf_{s>0} \tilde{f}(s) \end{aligned}$$

where

$$\tilde{f}(s) = \max(x, s)^{1-\alpha} \max(y, s)^{\alpha} s^{-\beta}$$

and

$$x = M_1/M_3, \; y = M_2/M_4.$$

Replacing the function f of [7, p. 380] by $\tilde{f}$ defined above and repeating the same arguments as in [7] the result follows.

Next we shall work out $D_{\alpha,\beta}(M_1, ..., M_N)$ in the general case of any convex polygon. The outcome will show the common geometrical base to Examples 1.5 and 1.6. We start with an auxiliary result.

LEMMA 1.7. *Let $Q_i = (x_i, y_i)$ for $i = 1, 2, 3$ be affinely independent in $\mathbb{R}^2$. For every positive real numbers M_1, M_2, M_3, there exist (unique) positive real numbers t_0, s_0 such that*

$$t_0^{x_1} s_0^{y_1} M_1 = t_0^{x_2} s_0^{y_2} M_2 = t_0^{x_3} s_0^{y_3} M_3. \tag{4}$$

Moreover, if

$$(\alpha, \beta) = \sum_{i=1}^{3} c_i Q_i \qquad \text{with} \quad \sum_{i=1}^{3} c_i = 1,$$

then

$$t_0^{x_1 - \alpha} s_0^{y_1 - \beta} M_1 = t_0^{x_2 - \alpha} s_0^{y_2 - \beta} M_2 = t_0^{x_3 - \alpha} s_0^{y_3 - \beta} M_3 = M_1^{c_1} M_2^{c_2} M_3^{c_3}.$$

Proof. By the affine independence of Q_1, Q_2, Q_3, the system of three equations in the unknown u, v, w given by

$$x_i u + y_i v - w = -\log M_i, \qquad i = 1, 2, 3$$

has determinant different from 0. Put $t_0 = e^u$ and $s_0 = e^v$, then t_0, s_0 satisfy (4) with the common value e^w.

Assume now

$$(\alpha, \beta) = \sum_{i=1}^{3} c_i Q_i \qquad \text{where} \quad \sum_{i=1}^{3} c_i = 1.$$

Let $\rho = t_0^{-\alpha} s_0^{-\beta} e^w$. Then

$$\rho = t_0^{x_i - \alpha} s_0^{y_i - \beta} M_i \qquad \text{for} \quad i = 1, 2, 3.$$

Raising to the power c_i we get

$$\rho^{c_i} = t_0^{(x_i - \alpha) c_i} s_0^{(y_i - \beta) c_i} M_i^{c_i}, \qquad i = 1, 2, 3,$$

and multiplying these equalities it follows that

$$\rho = \rho^{c_1 + c_2 + c_3} = M_1^{c_1} M_2^{c_2} M_3^{c_3}. \quad \blacksquare$$

Note that some numbers of c_1, c_2, c_3 might be negative. Lemma 1.7 shall be mainly used in case Q_1, Q_2, Q_3 are vertices P_i, P_k, P_r of Π and (α, β) is a convex combination of P_i, P_k, P_r. It will be useful to have a notation for the set of all such triples of vertices.

DEFINITION 1.8. Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon and let $(\alpha, \beta) \in \operatorname{Int} \Pi$. By $\mathscr{P}_{\alpha, \beta}$ we denote all those triples $\{i, k, r\}$ such that

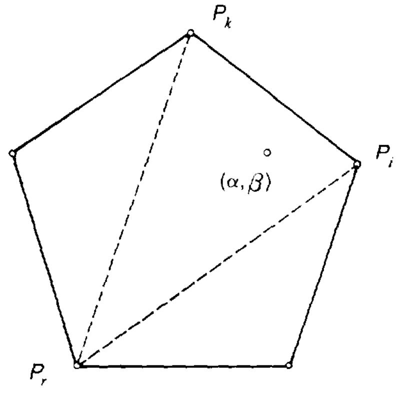

FIGURE 1.1

(α, β) belongs to the triangle with vertices P_i, P_k, P_r (see Fig. 1.1). In other words, $\{i, k, r\} \in \mathscr{P}_{\alpha, \beta}$ means that (α, β) can be written as a convex combination of P_i, P_k, P_r.

Now we are ready to determine $D_{\alpha, \beta}(M_1, ..., M_N)$.

THEOREM 1.9. *Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon with $P_j = (x_j, y_j)$ for $j = 1, ..., N$, let $(\alpha, \beta) \in \operatorname{Int} \Pi$ and let $D_{\alpha, \beta}$ and $\mathscr{P}_{\alpha, \beta}$ be as before. Then for any N-tuple of non-negative numbers $M_1, ..., M_N$, we have*

$$D_{\alpha, \beta}(M_1, ..., M_N) = \max\{M_i^{c_i} M_k^{c_k} M_r^{c_r} : \{i, k, r\} \in \mathscr{P}_{\alpha, \beta}\}.$$

Here (c_i, c_k, c_r) are the barycentric coordinates of (α, β) with respect to P_i, P_k, P_r.

Proof. Write

$$\mu = \max\{M_i^{c_i} M_k^{c_k} M_r^{c_r} : \{i, k, r\} \in \mathscr{P}_{\alpha, \beta}\}$$

and given any positive numbers t, s put

$$\varphi(t, s) = \max_{1 \leq j \leq N} \{t^{x_j - \alpha} s^{y_j - \beta} M_j\}$$

so

$$D_{\alpha, \beta}(M_1, ..., M_N) = \inf_{t > 0, s > 0} \{\varphi(t, s)\}.$$

For any $\varepsilon > 0$, we can find positive numbers t, s such that

$$\varphi(t, s) \leq D_{\alpha, \beta}(M_1, ..., M_N) + \varepsilon.$$

Thus

$$t^{x_j-\alpha}s^{y_j-\beta}M_j \leqslant D_{\alpha,\beta}(M_1, ..., M_N)+\varepsilon \qquad \text{for} \quad j=1, ..., N. \tag{5}$$

Take $\{i, k, r\} \in \mathscr{P}_{\alpha,\beta}$ and let (c_i, c_k, c_r) be the barycentric coordinates of (α, β) with respect to P_i, P_k, P_r. It follows from (5) that

$$t^{(x_j-\alpha)c_j}s^{(y_j-\beta)c_j}M_j^{c_j} \leqslant [D_{\alpha,\beta}(M_1, ..., M_N)+\varepsilon]^{c_j}, \qquad j=i, k, r.$$

Multiply together these inequalities to get

$$M_i^{c_i}M_k^{c_k}M_r^{c_r} \leqslant D_{\alpha,\beta}(M_1, ..., M_N)+\varepsilon.$$

Since $\{i, k\ r\}$ was taken arbitrarily in $\mathscr{P}_{\alpha,\beta}$ and $\varepsilon>0$ is also arbitrary, we conclude that

$$\mu \leqslant D_{\alpha,\beta}(M_1, ..., M_N).$$

Next, we prove the converse inequality. If, say, $\{1, 2, 3\} \in \mathscr{P}_{\alpha,\beta}$ and $\mu = M_1^{c_1}M_2^{c_2}M_3^{c_3}$, using Lemma 1.7, we can find positive numbers t_0, s_0 such that

$$t_0^{x_j-\alpha}s_0^{y_j-\beta}M_j=\mu \qquad \text{for} \quad j=1, 2, 3. \tag{6}$$

We claim that

$$\varphi(t_0, s_0)=\mu. \tag{7}$$

Indeed, if (7) does not hold, there must exist a fourth point, say P_4, such that

$$t_0^{x_4-\alpha}s_0^{y_4-\beta}M_4>\mu. \tag{8}$$

The extension of the segment joining P_4 and (α, β) in the direction of (α, β) must meet the side of the triangle $\overline{P_1P_2P_3}$ at a point Q which (obviously)

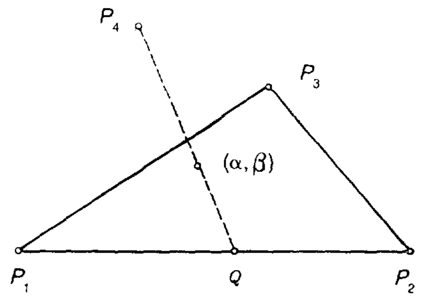

FIGURE 2

is a convex combination of two of P_1, P_2, P_3 (see Fig. 1.2). Say $Q \in \operatorname{Conv}(P_1, P_2)$.

Then (α, β) belongs to $\operatorname{Conv}(P_1, P_2, P_4)$; i.e., $\{1, 2, 4\} \in \mathscr{P}_{\alpha,\beta}$. Choose $0 \leqslant d_1, d_2, d_4 \leqslant 1$ with $d_1 + d_2 + d_4 = 1$ and $d_1 P_1 + d_2 P_2 + d_4 P_4 = (\alpha, \beta)$. Find also numbers $v_1, v_2, v_3 \in \mathbb{R}$ such that $v_1 + v_2 + v_3 = 1$ and $v_1 P_1 + v_2 P_2 + v_3 P_3 = P_4$. According to (6) and (8) we have that

$$\begin{aligned} M_1^{v_1} M_2^{v_2} M_3^{v_3} &= \prod_{j=1}^{3} [\mu^{v_j} t_0^{v_j(\alpha - x_j)} s_0^{v_j(\beta - y_j)}] \\ &= \mu t_0^{\alpha - x_4} s_0^{\beta - y_4} \\ &< M_4. \end{aligned}$$

On the other hand, it follows from the definition of μ that

$$\mu \geqslant M_1^{d_1} M_2^{d_2} M_4^{d_4}.$$

Hence

$$\mu > M_1^{d_1 + d_4 v_1} M_2^{d_2 + d_4 v_2} M_3^{d_4 v_3}. \tag{9}$$

But

$$\begin{aligned} (\alpha, \beta) &= d_1 P_1 + d_2 P_2 + d_4 P_4 \\ &= (d_1 + d_4 v_1) P_1 + (d_2 + d_4 v_2) P_2 + d_4 v_3 P_3. \end{aligned}$$

So, using Lemma 1.7 we get that

$$M_1^{d_1 + d_4 v_1} M_2^{d_2 + d_4 v_2} M_3^{d_4 v_3} = t_0^{x_1 - \alpha} s_0^{y_1 - \beta} M_1 = \mu$$

which contradicts (9).

This proves (7) and therefore

$$D_{\alpha,\beta}(M_1, ..., M_N) \leqslant \mu.$$

The proof is complete. ∎

Remark 1.10. Note that (7) implies that $D_{\alpha,\beta}$ is not only an infimum but a minimum

$$D_{\alpha,\beta}(M_1, ..., M_N) = \min_{t>0,\, s>0} [\max_{1 \leqslant j \leqslant N} \{t^{x_j - \alpha} s^{y_j - \beta} M_j\}].$$

We can recover Examples 1.5 and 1.6 as direct applications of this theorem. Moreover [Thm. 2.5 in 7] follows also easily from Theorem 1.8.

For later use, we also define the function

$$G_{\alpha,\beta}(M_1, ..., M_N) = \sup_{t>0,\, s>0} [\min_{1 \leqslant j \leqslant N} \{t^{x_j - \alpha} s^{y_j - \beta} M_j\}].$$

Observing that

$$G_{\alpha,\beta}(M_1, ..., M_N) = \left(D_{\alpha,\beta}\left(\frac{1}{M_1}, ..., \frac{1}{M_N}\right)\right)^{-1}$$

we get from Theorem 1.9.

THEOREM 1.11. *Let Π, (α, β), $\mathscr{P}_{\alpha,\beta}$ and $M_1, ..., M_N$ as in Theorem 1.9. Then*

$$G_{\alpha,\beta}(M_1, ..., M_N) = \min\{M_i^{c_i} M_k^{c_k} M_r^{c_r} : \{i, k, r\} \in \mathscr{P}_{\alpha,\beta}\},$$

where again (c_i, c_k, c_r) stands for the barycentric coordinates of (α, β) with respect to P_i, P_k, P_r.

2. INTERPOLATION OF WEIGHTED L_p-SPACES

Let (Ω, μ) be a measure space with σ-finite positive measure μ, and let A be a Banach space. If $w(x)$ is a positive μ-measurable function (*weight function*) and $1 \leqslant q \leqslant \infty$, we denote by $L_q(w; A)$ the Banach space of all strongly μ-measurable A-valued functions f having a finite norm

$$\|f\|_{L_q(w;A)} = \left(\int_\Omega \|w(x)\, f(x)\|_A^q \, d\mu\right)^{1/q}.$$

Next we study interpolation properties of weighted L_q-spaces. We restrict ourselves to the cases $q = 1$ and $q = \infty$. We shall derive formulae that involve the functions $D_{\alpha,\beta}$ and $G_{\alpha,\beta}$ of Section 1 but now acting on the weights.

DEFINITION 2.1. Let $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon with $P_j = (x_j, y_j)$, let $(\alpha, \beta) \in \operatorname{Int} \Pi$ and let $D_{\alpha,\beta}$ and $G_{\alpha,\beta}$ be the functions associated to them. If $w_1(x), ..., w_N(x)$ are weight functions on Ω we put

$$\hat{w}_{\alpha,\beta}(x) = D_{\alpha,\beta}(w_1(x), ..., w_N(x)) = \inf_{t>0,\, s>0} \left[\max_{1 \leqslant j \leqslant N} \{t^{x_j - \alpha} s^{y_j - \beta} w_j(x)\}\right]$$

and

$$\check{w}_{\alpha,\beta}(x) = G_{\alpha,\beta}(w_1(x), ..., w_N(x)) = \sup_{t>0,\, s>0} \left[\min_{1 \leqslant j \leqslant N} \{t^{x_j - \alpha} s^{y_j - \beta} w_j(x)\}\right].$$

As a direct consequence of Theorems 1.9 and 1.11 we have the following characterizations:

LEMMA 2.2. *Let $\mathscr{P}_{\alpha,\beta}$ be the set of triples associated to Π and (α,β). It holds*

$$\hat{w}_{\alpha,\beta}(x)=\max\{w_i^{c_i}(x)\,w_k^{c_k}(x)\,w_r^{c_r}(x):\{i,k,r\}\in\mathscr{P}_{\alpha,\beta}\}$$
$$\check{w}_{\alpha,\beta}(x)=\min\{w_i^{c_i}(x)\,w_k^{c_k}(x)\,w_r^{c_r}(x):\{i,k,r\}\in\mathscr{P}_{\alpha,\beta}\},$$

where (c_i, c_k, c_r) stands for the barycentric coordinates of (α,β) with respect to P_i, P_k, P_r.

We start by describing the space obtained applying the K-method to an L_∞-tuple.

THEOREM 2.3. *We have*

$$(L_\infty(w_1;A),\ldots,L_\infty(w_N;A))_{(\alpha,\beta),\infty;K}=L_\infty(\check{w}_{\alpha,\beta};A).$$

Proof. Assume that $f=\sum_{j=1}^N f_j$ where $f_j\in L_\infty(w_j;A)$. Given any $t>0$, $s>0$ and any $x\in\Omega$ one has

$$\begin{aligned}\sum_{j=1}^N t^{x_j}s^{y_j}\|f_j\|_{L_\infty(w_j;A)}&\geqslant\sum_{j=1}^N t^{x_j}s^{y_j}w_j(x)\,\|f_j(x)\|_A\\&\geqslant\min_{1\leqslant j\leqslant N}\{t^{x_j}s^{y_j}w_j(x)\}\sum_{j=1}^N\|f_j(x)\|_A\\&\geqslant\min_{1\leqslant j\leqslant N}\{t^{x_j}s^{y_j}w_j(x)\}\,\|f(x)\|_A.\end{aligned}$$

Hence

$$\min_{1\leqslant j\leqslant N}\{t^{x_j-\alpha}s^{y_j-\beta}w_j(x)\}\,\|f(x)\|_A\leqslant t^{-\alpha}s^{-\beta}K(t,s;f).$$

This shows that $(L_\infty(w_1;A),\ldots,L_\infty(w_N;A))_{(\alpha,\beta),\infty;K}$ is continuously embedded in $L_\infty(\check{w}_{\alpha,\beta};A)$.

To check the converse inclusion let $f\in L_\infty(\check{w}_{\alpha,\beta};A)$ and take any $t>0$, $s>0$. For $1\leqslant k\leqslant N$ write

$$A_k=\{x\in\Omega: t^{x_k}s^{y_k}w_k(x)\,\|f(x)\|_A=\min_{1\leqslant j\leqslant N}\{t^{x_j}s^{y_j}w_j(x)\,\|f(x)\|_A\}\}.$$

Define $\Gamma_1=A_1$, $\Gamma_k=A_k\setminus\bigcup_{1\leqslant j<k}A_j$ for $k=2,\ldots,N$, and set

$$f_j=\chi_{\Gamma_j}f\qquad\text{for}\quad j=1,\ldots,N,$$

where χ_{Γ_j} stands for the characteristic function of the set Γ_j. Clearly

$$f=\sum_{j=1}^N f_j$$

with $f_j \in L_\infty(w_j; A)$ because $f \in L_\infty(\check{w}_{\alpha,\beta}; A)$. Thus

$$\begin{aligned} t^{-\alpha}s^{-\beta}K(t, s; f) &\leqslant t^{-\alpha}s^{-\beta} \sum_{j=1}^{N} t^{x_j}s^{y_j} \|f_j\|_{L_\infty(w_j; A)} \\ &= \sum_{j=1}^{N} \sup_{x \in \Omega} \left[t^{x_j-\alpha}s^{y_j-\beta} w_j(x)\, \chi_{\Gamma_j}(x)\, \|f(x)\|_A \right] \\ &\leqslant N \sup_{x \in \Omega} \left[\min_{1 \leqslant j \leqslant N} \{ t^{x_j-\alpha}s^{y_j-\beta} w_j(x)\, \|f(x)\|_A \} \right] \\ &\leqslant N \|f\|_{L_\infty(\check{w}_{\alpha,\beta}; A)} \end{aligned}$$

which completes the proof. ∎

Let us go back for a moment to the question of how the choice of the polygon influences the resulting interpolation spaces (see Lemma 1.3). As an application of Theorem 2.3 we show next two polygons that generate different interpolation spaces.

EXAMPLE 2.4. Let $\{l_\infty, l_\infty(2^{-m}), l_\infty(2^{-n}), l_\infty(2^{-m-n})\}$ be the 4-tuple of scalar weighted l_∞-spaces over $\mathbb{Z} \times \mathbb{Z}$ where the weight functions are

$$w_1(m, n) = 1, \qquad w_2(m, n) = 2^{-m}, \qquad w_3(m, n) = 2^{-n}, \qquad w_4(m, n) = 2^{-m-n}.$$

Let Π be the polygon $\{(0, 0), (1, 0), (0, 1), (1/2, 1)\}$ and take the interior point $(1/3, 2/3)$. According to Theorem 2.3 we have

$$(l_\infty, l_\infty(2^{-m}), l_\infty(2^{-n}), l_\infty(2^{-m-n}))_{(1/3,\, 2/3),\, \infty;\, K} = l_\infty(w)$$

where

$$w(m, n) = 2^{-2n/3 - \max\{m/3,\, 2m/3\}}.$$

Take now Π equal to the unit square $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$ and any interior point $(\alpha, \beta) \in \operatorname{Int} \Pi$. Applying again Theorem 2.3 (or [7, Thm. 3.1]) the resulting interpolation space is

$$(l_\infty, l_\infty(2^{-m}), l_\infty(2^{-n}), l_\infty(2^{-m-n}))^F_{(\alpha,\beta),\, \infty;\, K} = l_\infty(2^{-m\alpha - n\beta}).$$

Hence $l_\infty(w)$ cannot be obtained by using the interpolation method associated to the square for any choice of the interior point (α, β).

Yet talking about the influence of the polygon on the resulting interpolation functors, observe that for the case of the square the definition of Fernandez spaces implies the following property of symmetry

$$(A_1, A_2, A_3, A_4)^F_{(\alpha,\beta),\, q;\, K} = (A_1, A_3, A_2, A_4)^F_{(\beta,\alpha),\, q;\, K}.$$

But if we interpolate by the methods associated to the polygon $\Pi=\{(0,0),(1,0),(0,1),(1/2,1)\}$, this property does not hold in general. Indeed, consider the 4-tuple

$$\left\{l_\infty\left(\frac{1}{n}\right), l_\infty\left(\frac{1}{\sqrt{n}}\right), l_\infty\left(\frac{1}{n}\right), l_\infty\left(\frac{1}{n}\right)\right\},$$

where we are now working over $\mathbb{N}$, the set of positive integers. Using Theorem 2.3 we get

$$\begin{aligned}
&\left(l_\infty\left(\frac{1}{n}\right), l_\infty\left(\frac{1}{\sqrt{n}}\right), l_\infty\left(\frac{1}{n}\right), l_\infty\left(\frac{1}{n}\right)\right)_{(1/2,1/2),\infty;K}\\
&=l_\infty\left(\frac{1}{n^{7/8}}\right)\neq l_\infty\left(\frac{1}{n}\right)\\
&=\left(l_\infty\left(\frac{1}{n}\right), l_\infty\left(\frac{1}{n}\right), l_\infty\left(\frac{1}{\sqrt{n}}\right), l_\infty\left(\frac{1}{n}\right)\right)_{(1/2,1/2),\infty;K}.
\end{aligned}$$

Next we consider the "dual" situation of Theorem 2.3: a J-interpolation formula for L_1-tuples. Then the relevant weight is $\hat{w}_{\alpha,\beta}$.

First note that given any N-tuple $\bar{A}=\{A_1,\dots,A_N\}$ the J-space $\bar{A}_{(\alpha,\beta),q;J}$ can be described in a discrete way, using sums instead of integrals. Namely, $\bar{A}_{(\alpha,\beta),q;J}$ coincides with the collection of all $a\in\Sigma(\bar{A})$ which can be represented as

$$a=\sum_{(m,n)\in\mathbb{Z}^2}u_{m,n}\qquad(\text{convergence in }\Sigma(\bar{A}))$$

where $u_{m,n}\in\Delta(\bar{A})$ and

$$\left[\sum_{(m,n)\in\mathbb{Z}^2}(2^{-m\alpha-n\beta}J(2^m,2^n;u_{m,n}))^q\right]^{1/q}<\infty.$$

Moreover, the norm $\|a\|_{(\alpha,\beta),q;J}$ is equivalent to

$$\inf\left\{\left[\sum_{(m,n)\in\mathbb{Z}^2}(2^{-m\alpha-n\beta}J(2^m,2^n;u_{m,n}))^q\right]^{1/q}\right\},$$

where the infimum is extended over all representations $(u_{m,n})$ of a as above. Subsequently, we also denote the discrete norm by $\|\ \|_{(\alpha,\beta),q;J}$. This, however, will produce no confusion.

THEOREM 2.5. *We have*

$$(L_1(w_1; A), ..., L_1(w_N; A))_{(\alpha, \beta), 1; J} = L_1(\hat{w}_{\alpha, \beta}; A).$$

Proof. Take $f \in (L_1(w_1; A), ..., L_1(w_N; A))_{(\alpha, \beta), 1; J}$ and let $\varepsilon > 0$. Using the discrete representation of J-spaces, we can find a sequence $(u_{m,n}) \subset \bigcap_{j=1}^{N} L_1(w_j; A)$ such that

$$f = \sum u_{m,n}$$

and

$$\sum_{(m,n) \in \mathbb{Z}^2} 2^{-m\alpha - n\beta} J(2^m, 2^n; u_{m,n}) \leqslant (1 + \varepsilon) \|f\|_{(\alpha, \beta), 1; J}.$$

Then

$$\begin{aligned}
\|f\|_{L_1(\hat{w}_{\alpha,\beta}; A)} &= \int_\Omega \hat{w}_{\alpha,\beta}(x) \left\| \sum_{(m,n) \in \mathbb{Z}^2} u_{m,n}(x) \right\|_A d\mu \\
&\leqslant \sum_{(m,n) \in \mathbb{Z}^2} \int_\Omega \max_{1 \leqslant j \leqslant N} \{2^{m(x_j - \alpha) + n(y_j - \beta)} w_j(x)\} \|u_{m,n}(x)\|_A \, d\mu \\
&\leqslant \sum_{(m,n) \in \mathbb{Z}^2} 2^{-m\alpha - n\beta} \left(\sum_{j=1}^{N} 2^{mx_j + ny_j} \int_\Omega w_j(x) \|u_{m,n}(x)\|_A \, d\mu \right) \\
&\leqslant N \sum_{(m,n) \in \mathbb{Z}^2} 2^{-m\alpha - n\beta} J(2^m, 2^n; u_{m,n}) \\
&\leqslant N(1 + \varepsilon) \|f\|_{(\alpha, \beta), 1; J}.
\end{aligned}$$

Conversely, let f be a simple function. Without loss of generality, we may assume that the weight functions w_j are discrete valued. Then f can be written as

$$f = \sum_{r=1}^{\infty} a_r \chi_{\Gamma_r},$$

where $a_r \in A$, the sets Γ_r are disjoint with finite μ-measure and the weights w_j are all constant on each Γ_r. It follows from Lemma 2.2 that $\hat{w}_{\alpha, \beta}$ is also constant on each Γ_r. Moreover according to Remark 1.10, for each r there are $t_r, s_r > 0$ such that

$$\max_{1 \leqslant j \leqslant N} \{t_r^{x_j - \alpha} s_r^{y_j - \beta} w_j(x)\} = \hat{w}_{\alpha, \beta}(x) \qquad \text{for all} \quad x \in \Gamma_r.$$

Given $x \in \Omega$, define (t_x, s_x) by

$$(t_x, s_x) = \begin{cases} (t_r, s_r) & \text{if } x \in \Gamma_r \\ (0, 0) & \text{if } x \in \Omega \setminus \bigcup_{r=1}^{\infty} \Gamma_r. \end{cases}$$

Next, for $(m, n) \in \mathbb{Z}^2$, put

$$A_{m,n} = \{x \in \Omega : 2^m \leqslant t_x < 2^{m+1} \text{ and } 2^n \leqslant s_x < 2^{n+1}\}$$

and

$$u_{m,n} = \chi_{A_{m,n}} f.$$

In this way we get a representation of f as

$$f = \sum_{(m,n) \in \mathbb{Z}^2} u_{m,n} \qquad \text{with} \quad u_{m,n} \in \bigcap_{j=1}^{N} L_1(w_j; A).$$

Therefore

$$\begin{aligned} \|f\|_{(\alpha,\beta),1;J} &\leqslant \sum_{(m,n) \in \mathbb{Z}^2} 2^{-m\alpha - n\beta} J(2^m, 2^n; u_{m,n}) \\ &= \sum_{(m,n) \in \mathbb{Z}^2} \max_{1 \leqslant j \leqslant N} \left\{ \int_\Omega 2^{m(x_j - \alpha) + n(y_j - \beta)} w_j(x) \, \|u_{m,n}(x)\|_A \, d\mu \right\} \\ &\leqslant C \sum_{(m,n) \in \mathbb{Z}^2} \max_{1 \leqslant j \leqslant N} \left(\int_{A_{m,n}} t_x^{(x_j - \alpha)} s_x^{(y_j - \beta)} w_j(x) \, \|f(x)\|_A \, d\mu \right) \\ &= C \sum_{(m,n) \in \mathbb{Z}^2} \int_{A_{m,n}} \hat{w}_{\alpha,\beta}(x) \, \|f(x)\|_A \, d\mu \\ &= C \|f\|_{L_1(\hat{w}_{\alpha,\beta}; A)}. \end{aligned}$$

Since simple functions are dense in $L_1(\hat{w}_{\alpha,\beta}; A)$, the result follows. ∎

Let us write down some other concrete cases of these theorems.

COROLLARY 2.6. *Let Π be the unit square and $\alpha = \beta = 1/2$. Then*

$$\check{w}_{1/2,1/2}(x) = \min\{\sqrt{w_1(x)\, w_4(x)}, \sqrt{w_2(x)\, w_3(x)}\}$$

while

$$\hat{w}_{1/2,1/2}(x) = \max\{\sqrt{w_1(x)\, w_4(x)}, \sqrt{w_2(x)\, w_3(x)}\}.$$

In particular, if $w_1 = w_4$ and $w_2 = w_3$, then

$$(L_\infty(w_1; A), L_\infty(w_2; A), L_\infty(w_2; A), L_\infty(w_1; A))_{(1/2, 1/2), \infty; K}$$
$$= L_\infty(w_1; A) + L_\infty(w_2; A)$$

and

$$(L_1(w_1; A), L_1(w_2; A), L_1(w_2; A), L_1(w_1; A))_{(1/2, 1/2), 1; J}$$
$$= L_1(w_1; A) \cap L_1(w_2; A).$$

The following formula refers to K-interpolation of L_1-tuples.

THEOREM 2.7. *We have*

$$(L_1(w_1; A), ..., L_1(w_N; A))_{(\alpha, \beta), 1; K} = L_1(\eta; A),$$

where the weight η is defined by

$$\eta(x) = \int_0^\infty \int_0^\infty \min_{1 \leq j \leq N} \{t^{x_j - \alpha} s^{y_j - \beta} w_j(x)\} \frac{dt}{t} \frac{ds}{s}.$$

Proof. It is not too difficult to see that

$$K(t, s; f) = \int_\Omega \min_{1 \leq j \leq N} \{t^{x_j} s^{y_j} w_j(x)\} \|f(x)\|_A \, d\mu.$$

Whence

$$\begin{aligned}
\|f\|_{(\alpha, \beta), 1; K} &= \int_0^\infty \int_0^\infty t^{-\alpha} s^{-\beta} K(t, s; f) \frac{dt}{t} \frac{ds}{s} \\
&= \int_\Omega \int_0^\infty \int_0^\infty \min_{1 \leq j \leq N} \{t^{x_j - \alpha} s^{y_j - \beta} w_j(x)\} \|f(x)\|_A \frac{dt}{t} \frac{ds}{s} \, d\mu \\
&= \int_\Omega \eta(x) \|f(x)\|_A \, d\mu \\
&= \|f\|_{L_1(\eta; A)}. \quad \blacksquare
\end{aligned}$$

As an application of Theorems 2.5 and 2.7 we shall show another point where the theory of Fernandez spaces (Example 1.2) differs from the theory of Sparr spaces (Example 1.1). A quick look at [10 or 11] might suggest that Fernandez spaces have a theory parallel to Sparr's theory [16]. However, the fact that parameters t and s appear together in the K- and J-functionals of Fernandez while they do not in Sparr functionals, causes significant differences between their theories. A first hint in this direction is the behaviour of norms of interpolated operators (see Examples 1.5 and 1.6). Next we describe another difference.

Sparr proved in [16, Thms. 8.1 and 8.3] (see also Prop. 8.1) that for any 3-tuple of weighted L_1-spaces it holds

$$(L_1(w_1; A), L_1(w_2; A), L_1(w_3; A))^S_{(\alpha,\beta),1;J}$$
$$= (L_1(w_1; A), L_1(w_2; A), L_1(w_3; A))^S_{(\alpha,\beta),1;K},$$

the resulting space being

$$L_1(w_1^{1-\alpha-\beta} w_2^{\alpha} w_3^{\beta}; A).$$

However, as the following example shows, K- and J-Fernandez spaces might not agree on an L_1-tuple.

EXAMPLE 2.8. Let Π be the unit square $\{(0,0), (1,0), (0,1), (1,1)\}$, and let $\alpha = \beta = 1/2$. For $n \in \mathbb{N}$, put

$$w_1(n) = w_4(n) = \frac{1}{\sqrt{n}}, \qquad w_2(n) = w_3(n) = \frac{1}{n}$$

and consider the following 4-tuple of scalar weighted sequence spaces over $\mathbb{N}$

$$\bar{X} = \left\{ l_1\left(\frac{1}{\sqrt{n}}\right), l_1\left(\frac{1}{n}\right), l_1\left(\frac{1}{n}\right), l_1\left(\frac{1}{\sqrt{n}}\right) \right\}.$$

According to Corollary 2.6, we have

$$\bar{X}_{(1/2,1/2),1;J} = l_1\left(\frac{1}{\sqrt{n}}\right).$$

On the other hand, Theorem 2.7 gives that

$$\bar{X}_{(1/2,1/2),1;K} = l_1(\eta(n)),$$

where

$$\eta(n) = \int_0^\infty \int_0^\infty \min\left\{ t^{-1/2} s^{-1/2} \frac{1}{\sqrt{n}}, t^{1/2} s^{-1/2} \frac{1}{n}, t^{-1/2} s^{1/2} \frac{1}{n}, t^{1/2} s^{1/2} \frac{1}{\sqrt{n}} \right\} \frac{dt}{t} \frac{ds}{s}.$$

Let us work out $\eta(n)$. Writing the integral as

$$\eta(n)=\int_0^\infty\int_0^\infty \min\Big[\min\{t^{-1/2}s^{-1/2},\, t^{1/2}s^{1/2}\}\,\frac{1}{\sqrt{n}},$$
$$\min\{t^{1/2}s^{-1/2},\, t^{-1/2}s^{1/2}\}\,\frac{1}{n}\Big]\frac{dt}{t}\frac{ds}{s}$$

and making the change of variables

$$t=e^{u+v},\qquad s=e^{u-v}$$

we get

$$\begin{aligned}\eta(n)&=2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\min\left\{\frac{e^{-|u|}}{\sqrt{n}},\frac{e^{-|v|}}{n}\right\}du\,dv\\&=8\int_0^\infty\int_0^\infty\min\left\{\frac{e^{-u}}{\sqrt{n}},\frac{e^{-v}}{n}\right\}du\,dv\\&=\frac{16}{n}+4\,\frac{\log n}{n}.\end{aligned}$$

We see then that $\eta(n)$ is equivalent to $\log n/n$. Consequently

$$\bar{X}_{(1/2,\,1/2),\,1;\,K}=l_1\left(\frac{\log n}{n}\right)\neq l_1\left(\frac{1}{\sqrt{n}}\right)=\bar{X}_{(1/2,\,1/2),\,1;\,J}.$$

Let us go back to the general situation but assuming this time certain relationships on the weights.

THEOREM 2.9. *Let $\Pi=\overline{P_1\cdots P_N}$ be a convex polygon with $P_j=(x_j,y_j)$ and let $(\alpha,\beta)\in\operatorname{Int}\Pi$. If w_1, w_2, w_3 are weight functions on Ω, we define*

$$\begin{aligned}\tilde{w}_j(x)&=w_1^{1-x_j-y_j}(x)\,w_2^{x_j}(x)\,w_3^{y_j}(x),\qquad j=1,\dots,N,\\ w(x)&=w_1^{1-\alpha-\beta}(x)\,w_2^{\alpha}(x)\,w_3^{\beta}(x)\end{aligned}$$

and given any Banach N-tuple $\bar{A}=\{A_1,\dots,A_N\}$, we form the vector valued weighted spaces $L_1(\tilde{w}_j;A_j)$ $(1\leqslant j\leqslant N)$. Then

$$(L_1(\tilde{w}_1;A_1),\dots,L_1(\tilde{w}_N;A_N))_{(\alpha,\beta),1;K}=L_1(w;\bar{A}_{(\alpha,\beta),1;K}).$$

Proof. Denote by $K(t,s;\cdot)$ the K-functional with respect to the N-tuple $\{L_1(\tilde{w}_j;A_j)\}_{j=1}^N$, while $K'(t,s;\cdot)$ refers to $\{A_j\}_{j=1}^N$.

Arguing as in [16], Lemma 8.4, one can check that

$$K(t, s; f) = \inf\left\{ \sum_{j=1}^{N} \int_\Omega w_1(x) \left(\frac{tw_2(x)}{w_1(x)}\right)^{x_j} \left(\frac{sw_3(x)}{w_1(x)}\right)^{y_j} \|f_j(x)\|_{A_j}\, d\mu : f = \sum_{j=1}^{N} f_j \right\}$$

$$= \int_\Omega w_1(x)\, K'\left(\frac{tw_2(x)}{w_1(x)}, \frac{sw_3(x)}{w_1(x)}; f(x)\right) d\mu.$$

Whence

$$\|f\|_{(\alpha,\beta),1;K}$$

$$= \int_\Omega w_1^{1-\alpha-\beta}(x)\, w_2^\alpha(x)\, w_3^\beta(x)$$

$$\times \int_0^\infty \int_0^\infty \left(\frac{tw_2(x)}{w_1(x)}\right)^{-\alpha} \left(\frac{sw_3(x)}{w_1(x)}\right)^{-\beta} K'\left(\frac{tw_2(x)}{w_1(x)}, \frac{sw_3(x)}{w_1(x)}; f(x)\right) \frac{dt}{t}\frac{ds}{s}\, d\mu$$

$$= \int_\Omega w(x)\, \|f(x)\|_{(\alpha,\beta),1;K}\, d\mu$$

$$= \|f\|_{L_1(w;\, \bar{A}_{(\alpha,\beta),1;K})}. \quad \blacksquare$$

A similar formula holds for the J-method. We write this time $J(t, s; \cdot)$ for the J-functional associated to $\{L_1(\tilde{w}_j; A_j)\}_{j=1}^N$ and $J'(t, s; \cdot)$ for the corresponding one to $\{A_j\}_{j=1}^N$.

Theorem 2.10. *Under the same assumptions as in Theorem* 2.9, *we have*

$$(L_1(\tilde{w}_1; A_1), ..., L_1(\tilde{w}_N; A_N))_{(\alpha,\beta),1;J} = L_1(w; \bar{A}_{(\alpha,\beta),1;J}).$$

Proof. Using the discrete characterizations of J-spaces, one can verify that $\Delta(\bar{A})$ is dense in $\bar{A}_{(\alpha,\beta),1;J}$. Then it is not hard to verify that simple functions f of the form

$$f = \sum_r a_r \chi_{\Gamma_r} \qquad \text{(finite sum)}, \tag{$*$}$$

where $a_r \in \Delta(\bar{A})$, $\mu(\Gamma_r) < \infty$ and $\sup_{x \in \Gamma_r} \sum_{j=1}^N \tilde{w}_j(x) < \infty$ are dense in both spaces appearing in the statement. So in what follows, we assume that f has the form $(*)$.

For each r, find a representation

$$a_r = \int_0^\infty \int_0^\infty v_r(t, s) \frac{dt}{t}\frac{ds}{s}$$

with

$$\int_0^\infty \int_0^\infty t^{-\alpha} s^{-\beta} J'(t, s; v_r(t, s)) \frac{dt}{t} \frac{ds}{s} \leqslant (1+\varepsilon) \|a_r\|_{(\alpha,\beta),1;J}.$$

Setting

$$u(t, s)(x) = \sum_r v_r \left(\frac{t w_2(x)}{w_1(x)}, \frac{s w_3(x)}{w_1(x)} \right) \chi_{\Gamma_r}(x)$$

we obtain a representation of f,

$$f = \int_0^\infty \int_0^\infty u(t, s) \frac{dt}{t} \frac{ds}{s}.$$

Since

$$\begin{aligned} &J(t, s; u(t, s)) \\ &\quad \leqslant \sum_{j=1}^{N} \int_\Omega \sum_r w_1(x) \left(\frac{t w_2(x)}{w_1(x)} \right)^{x_j} \left(\frac{s w_3(x)}{w_1(x)} \right)^{y_j} \left\| v_r \left(\frac{t w_2(x)}{w_1(x)}, \frac{s w_3(x)}{w_1(x)} \right) \right\|_{A_j} \chi_{\Gamma_r}(x)\, d\mu \\ &\quad \leqslant N \int_\Omega \sum_r w_1(x)\, J' \left(\frac{t w_2(x)}{w_1(x)}, \frac{s w_3(x)}{w_1(x)}; v_r \left(\frac{t w_2(x)}{w_1(x)}, \frac{s w_3(x)}{w_1(x)} \right) \right) \chi_{\Gamma_r}(x)\, d\mu \end{aligned}$$

we derive

$$\begin{aligned} \|f\|_{(\alpha,\beta),1;J} &\leqslant \int_0^\infty \int_0^\infty t^{-\alpha} s^{-\beta} J(t, s; u(t, s)) \frac{dt}{t} \frac{ds}{s} \\ &\leqslant N \int_\Omega \sum_r w_1^{1-\alpha-\beta}(x)\, w_2^{\alpha}(x)\, w_3^{\beta}(x)\, \chi_{\Gamma_r}(x) \\ &\quad \times \int_0^\infty \int_0^\infty \left(\frac{t w_2(x)}{w_1(x)} \right)^{-\alpha} \left(\frac{s w_3(x)}{w_1(x)} \right)^{-\beta} \\ &\quad \times J' \left(\frac{t w_2(x)}{w_1(x)}, \frac{s w_3(x)}{w_1(x)}; v_r \left(\frac{t w_2(x)}{w_1(x)}, \frac{s w_3(x)}{w_1(x)} \right) \right) \frac{dt}{t} \frac{ds}{s}\, d\mu \\ &\leqslant N(1+\varepsilon) \int_\Omega \sum_r w(x)\, \chi_{\Gamma_r}(x)\, \|a_r\|_{(\alpha,\beta),1;J}\, d\mu \\ &\leqslant (1+\varepsilon)\, N \int_\Omega w(x)\, \|f(x)\|_{(\alpha,\beta),1;J}\, d\mu \\ &= (1+\varepsilon)\, N\, \|f\|_{L_1(w;\, \bar{A}_{(\alpha,\beta),1;J})}. \end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ we get that

$$\|f\|_{(\alpha,\beta),1;J} \leqslant N \|f\|_{L_1(w;\bar{A}_{(\alpha,\beta),1;J})}.$$

To prove the converse inequality, suppose that

$$f = \int_0^\infty \int_0^\infty u(t,s) \frac{dt}{t} \frac{ds}{s}$$

with

$$\int_0^\infty \int_0^\infty t^{-\alpha} s^{-\beta} J(t,s;u(t,s)) \frac{dt}{t} \frac{ds}{s} \leqslant (1+\varepsilon) \|f\|_{(\alpha,\beta),1;J}.$$

Then we have

$$\begin{aligned}
\|f\|_{L_1(w;\bar{A}_{(\alpha,\beta),1;J})} &= \int_\Omega w(x) \|f(x)\|_{(\alpha,\beta),1;J}\, d\mu \\
&\leqslant \int_\Omega w_1^{1-\alpha-\beta}(x)\, w_2^\alpha(x)\, w_3^\beta(x) \\
&\quad \times \int_0^\infty \int_0^\infty \left(\frac{tw_2(x)}{w_1(x)}\right)^{-\alpha} \left(\frac{sw_3(x)}{w_1(x)}\right)^{-\beta} \\
&\quad \times J'\left(\frac{tw_2(x)}{w_1(x)}, \frac{sw_3(x)}{w_1(x)}; u(t,s)(x)\right) \frac{dt}{t} \frac{ds}{s}\, d\mu \\
&\leqslant \int_0^\infty \int_0^\infty t^{-\alpha} s^{-\beta} \sum_{j=1}^N t^{x_j} s^{y_j} \int_\Omega \tilde{w}_j(x) \|u(t,s)(x)\|_{A_j}\, d\mu \frac{dt}{t} \frac{ds}{s} \\
&\leqslant N \int_0^\infty \int_0^\infty t^{-\alpha} s^{-\beta} J(t,s;u(t,s)) \frac{dt}{t} \frac{ds}{s} \\
&\leqslant (1+\varepsilon) N \|f\|_{(\alpha,\beta),1;J}.
\end{aligned}$$

The proof is complete. ∎

Remark 2.11. Note that in Theorems 2.9 and 2.10 we have

$$\check{w}_{\alpha,\beta} = \hat{w}_{\alpha,\beta} = w.$$

In the case of arbitrary weights $w_1, ..., w_N$, similar arguments to those used before prove

$$\begin{aligned}
L_1(\hat{w}_{\alpha,\beta}; \bar{A}_{(\alpha,\beta),1;K}) &\hookrightarrow (L_1(w_1;A_1), ..., L_1(w_N;A_N))_{(\alpha,\beta),1;K} \\
&\hookrightarrow L_1(\check{w}_{\alpha,\beta}; \bar{A}_{(\alpha,\beta),1;K})
\end{aligned}$$

with analogous embeddings holding for the J-spaces.

3. J–K Estimates

As we have seen in Section 1, in general we cannot estimate the norms of operators interpolated by the J- or K-method in terms of the product of positive powers of the norms of all restrictions $T: A_j \to B_j$ $(1 \leqslant j \leqslant N)$. The fact that the estimate is by the maximum of products of powers of three norms causes a number of problems in developing the theory.

Sometimes one can come out of the difficulty by imposing a certain (geometrical) condition on the polygon (see [7, Sect. 5]). Another possibility is to consider operators from a J-space into a K-space (see [5, Sect. 4, or 4, Sect. 3]), then the norm can be estimated by

$$\|T\|_{\bar{A}_{(\alpha,\beta),q;J},\,\bar{B}_{(\alpha,\beta),q;K}} \leqslant \gamma\big(\min_{1\leqslant j\leqslant N}\{\|T\|_{A_j,B_j}\}\big)^{\tau}\big(\max_{1\leqslant j\leqslant N}\{\|T\|_{A_j,B_j}\}\big)^{1-\tau}, \tag{10}$$

where $\gamma>0$ and $0<\tau<1$ are constants depending only on Π and (α,β).

Inequality (10) was established by Cobos *et al.* [5, Thm. 4.3], by means of direct computations. In what follows, we shall develop a completely different approach based on the relationship between K- and J-methods and Sparr spaces defined by using $N-1$ parameters. The new approach will allow a better understanding of estimate (10) and also will give interesting results referring to the coincidence of K- and J-methods.

We begin by reviewing Sparr constructions. If $\bar{t}=(t_1,...,t_N)$ and $\bar{s}=(s_1,...,s_N)$ are N-tuple of positive numbers, we set

$$\bar{t}\bar{s}=(t_1s_1,...,t_Ns_N), \qquad 2^{\bar{t}}=(2^{t_1},...,2^{t_N}), \qquad |\bar{t}|=t_1\cdots t_N.$$

By $\bar{v}=(v_1,...,v_{N-1})$ we mean an $(N-1)$-tuple of integer numbers. Associated to $\bar{v}$ we have the N-tuple $\hat{v}=(0,v_1,...,v_{N-1})$.

Let $\bar{A}=\{A_j\}_{j=1}^N$ be any Banach N-tuple. The relevant K- and J-functionals are now

$$K_S(\bar{t},a)=\inf\left\{\sum_{j=1}^N t_j\|a_j\|_{A_j} : a=\sum_{j=1}^N a_j,\ a_j\in A_j\right\}$$

$$J_S(\bar{t},a)=\max_{1\leqslant j\leqslant N}\{t_j\|a\|_{A_j}\}.$$

Observe that parameters t_j do not appear combined in K_S nor J_S.

Assume that $1\leqslant q\leqslant\infty$ and that $\bar{\theta}=(\bar{\theta}_1,...,\bar{\theta}_N)$ is an N-tuple of positive numbers with $\sum_{j=1}^N\theta_j=1$. The space $\bar{A}^S_{\bar{\theta},q;K}$ is the collection of all those elements $a\in\Sigma(\bar{A})$ which have a finite norm

$$\|a\|^S_{\bar{\theta},q;K}=\left(\sum_{\bar{v}\in\mathbb{Z}^{N-1}}(|2^{-\hat{v}\bar{\theta}}|\,K_S(2^{\hat{v}},a))^q\right)^{1/q}$$

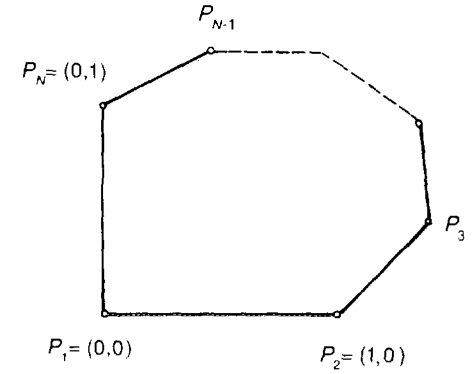

FIGURE 3.1

while $\bar{A}^S_{\bar{\theta},q;J}$ is formed by all those elements $a \in \Sigma(\bar{A})$ which have a representation of the form

$$a = \sum_{\bar{v} \in \mathbb{Z}^{N-1}} u_{\bar{v}} \qquad \text{(convergence in } \Sigma(\bar{A})\text{)}, \tag{11}$$

where $u_{\bar{v}} \in \Delta(\bar{A})$ and

$$\left(\sum_{\bar{v} \in \mathbb{Z}^{N-1}} (|2^{-\bar{v}\bar{\theta}}| \, J_S(2^{\bar{v}}, u_{\bar{v}}))^q \right)^{1/q} < \infty. \tag{12}$$

The norm $\|a\|^S_{\bar{\theta},q;J}$ of $\bar{A}^S_{\bar{\theta},q;J}$ is the infimum of the values of the sum (12) over all sequences (u_v) satisfying (11) and (12). Sparr spaces admit equivalent definitions in terms of integrals (i.e., continuous descriptions) but they will not be needed here.

Let again $\Pi = \overline{P_1 \cdots P_N}$ be a convex polygon with $P_j = (x_j, y_j)$, and let $(\alpha, \beta) \in \operatorname{Int} \Pi$. Taking into account Lemma 1.3, we may assume without loss of generality that $P_1 = (0, 0)$, $P_2 = (1, 0)$ and $P_N = (0, 1)$. In other words, Π has the form described in Fig. 3.1.

Find $0 < \theta_1, ..., \theta_N < 1$ with $\sum_{j=1}^N \theta_j = 1$ and $\sum_{j=1}^N \theta_j P_j = (\alpha, \beta)$, that is to say, $\bar{\theta} = (\theta_1, ..., \theta_N)$ are barycentric coordinates of (α, β) with respect to the vertices $P_1, ..., P_N$. Observe that if $N \geq 4$, such coordinates are not unique.

To compare Sparr constructions with K- and J-methods associated to Π, we shall need the discrete characterization of $\bar{A}_{(\alpha,\beta),q;J}$ as given in Section 2. The discrete representation of the K-space is

$$\bar{A}_{(\alpha,\beta),q;K} = \left\{ a \in \Sigma(\bar{A}) : \|a\|_{(\alpha,\beta),q;K} = \left(\sum_{(m,n) \in \mathbb{Z}^2} (2^{-m\alpha - n\beta} K(2^m, 2^n; a))^q \right)^{1/q} < \infty \right\}.$$

Subsequently, we use in our notation the scalar product $\langle \cdot, \cdot \rangle$ of $\mathbb{R}^2$ writing

$$mx_j + ny_j = \langle P_j, (m, n) \rangle$$

and by $[x]$ we mean the integer part of the real number x, i.e. the largest integer which is less than or equal to x.

The origin of the next result is [7, Thm. 1.4].

THEOREM 3.1. *Let $\bar{\theta} = (\theta_1, \ldots, \theta_N)$ be some barycentric coordinates of (α, β) with respect to the vertices $P_1, \ldots, P_N$. Then we have with continuous embeddings*

$$\bar{A}_{(\alpha, \beta), q; J} \hookrightarrow \bar{A}^S_{\bar{\theta}, q; J} \hookrightarrow \bar{A}^S_{\bar{\theta}, q; K} \hookrightarrow \bar{A}_{(\alpha, \beta), q; K}.$$

Proof. For $a \in \bar{A}^S_{\bar{\theta}, q; K}$ it holds

$$\begin{aligned}
\|a\|^S_{\bar{\theta}, q; K} &= \Big(\sum_{\bar{v} \in \mathbb{Z}^{N-1}} (|2^{-\bar{v}\bar{\theta}}| \, K_S(2^{\bar{v}}, a))^q \Big)^{1/q} \\
&\geqslant \Big(\sum_{(m, n) \in \mathbb{Z}^2} \Big(2^{-\theta_2 m - \theta_N n - \sum_{j=3}^{N-1} \theta_j [mx_j + ny_j]} \inf \Big\{ \|a_1\|_{A_1} + 2^m \, \|a_2\|_{A_2} \\
&\quad + 2^n \, \|a_N\|_{A_N} + \sum_{j=3}^{N-1} 2^{[mx_j + ny_j]} \, \|a_j\|_{A_j} \Big\} \Big)^q \Big)^{1/q} \\
&\geqslant \Big(\sum_{(m, n) \in \mathbb{Z}^2} \Big(2^{-\theta_2 m - \theta_N n - \sum_{j=3}^{N-1} \theta_j (mx_j + ny_j)} \inf \Big\{ \|a_1\|_{A_1} + 2^m \, \|a_2\|_{A_2} \\
&\quad + 2^n \, \|a_N\|_{A_N} + \sum_{j=3}^{N-1} \frac{1}{2} 2^{(mx_j + ny_j)} \, \|a_j\|_{A_j} \Big\} \Big)^q \Big)^{1/q} \\
&\geqslant \frac{1}{2} \Big(\sum_{(m, n) \in \mathbb{Z}^2} (2^{-\sum_{j=1}^{N} \theta_j \langle P_j, (m, n) \rangle} K(2^m, 2^n; a))^q \Big)^{1/q} \\
&= \frac{1}{2} \Big(\sum_{(m, n) \in \mathbb{Z}^2} (2^{-m\alpha - n\beta} K(2^m, 2^n; a))^q \Big)^{1/q} \\
&= \tfrac{1}{2} \|a\|_{(\alpha, \beta), q; K}.
\end{aligned}$$

On the other hand, if $a \in \bar{A}_{(\alpha, \beta), q; J}$ and

$$a = \sum_{(m, n) \in \mathbb{Z}^2} u_{m, n}$$

is a representation of a such that

$$\left(\sum_{(m,n) \in \mathbb{Z}^2} (2^{-m\alpha - n\beta} J(2^m, 2^n; u_{m,n}))^q \right)^{1/q} \leqslant (1+\varepsilon) \|a\|_{(\alpha,\beta),q;J}$$

then setting for $\bar{v} \in \mathbb{Z}^{N-1}$

$$\tilde{u}_{\bar{v}} = \begin{cases} u_{m,n} & \text{if } \begin{array}{l} v_1 = m,\ v_{N-1} = n, \\ v_j = [mx_{j+1} + ny_{j+1}] \quad (2 \leqslant j \leqslant N-2) \end{array} \\ 0 & \text{otherwise} \end{cases}$$

we obtain another representation of a, now in the form

$$a = \sum_{\bar{v} \in \mathbb{Z}^{N-1}} \tilde{u}_{\bar{v}}$$

and therefore

$$\begin{aligned} \|a\|^S_{\bar{\theta},q;J} &\leqslant \left(\sum_{\bar{v} \in \mathbb{Z}^{N-1}} (|2^{-\bar{v}\bar{\theta}}|\, J(2^{\bar{v}}, u_{\bar{v}}))^q \right)^{1/q} \\ &= \left(\sum_{(m,n) \in \mathbb{Z}^2} (2^{-\theta_2 m - \theta_N n - \sum_{j=3}^{N-1} \theta_j [mx_j + ny_j]} \right. \\ &\quad \times \max_{3 \leqslant j \leqslant N-1} \{ \|u_{m,n}\|_{A_1}, 2^m \|u_{m,n}\|_{A_2}, 2^n \|u_{m,n}\|_{A_N}, \\ &\quad \left. 2^{[mx_j + ny_j]} \|u_{m,n}\|_{A_j} \})^q \right)^{1/q} \\ &\leqslant \left(\sum_{(m,n) \in \mathbb{Z}^2} (2^{-\sum_{j=1}^N \theta_j \langle P_j, (m,n) \rangle + \sum_{j=3}^{N-1} \theta_j} J(2^m, 2^n; u_{m,n}))^q \right)^{1/q} \\ &\leqslant 2 \left(\sum_{(m,n) \in \mathbb{Z}^2} (2^{-m\alpha - n\beta} J(2^m, 2^n; u_{m,n}))^q \right)^{1/q} \\ &\leqslant 2(1+\varepsilon) \|a\|_{(\alpha,\beta),q;J}. \end{aligned}$$

The remaining embedding $\bar{A}^S_{\bar{\theta},q;J} \hookrightarrow \bar{A}^S_{\bar{\theta},q;K}$ was proved by Sparr [16], Prop. 5.1. ∎

We are in a position to establish the norm estimate.

THEOREM 3.2. *Let $\bar{\theta} = (\theta_1, ..., \theta_N)$ be some barycentric coordinates of (α, β) with respect to the vertices $P_1, ..., P_N$ of Π. There exists a constant*

$C>0$, depending only on $\bar{\theta}$, such that for any Banach N-tuples $\bar{A}=\{A_1, ..., A_N\}$, $\bar{B}=\{B_1, ..., B_N\}$ and any operator $T\colon \bar{A}\to\bar{B}$ we have

$$\|T\|_{\bar{A}_{(\alpha,\beta),q;J},\,\bar{B}_{(\alpha,\beta),q;K}} \leqslant C\prod_{j=1}^{N} M_j^{\theta_j},$$

where $M_j=\|T\|_{A_j,B_j}$ for $j=1, ..., N$. ∎

Proof. We recall that Sparr methods of parameters $\bar{\theta}, q$ are interpolation functors of exponent $\bar{\theta}$ (see [16, Sect. 4]); that is, the norm of the interpolated operator is less than or equal to

$$\prod_{j=1}^{N} M_j^{\theta_j}.$$

Combining this piece of information with Theorem 3.1 and the fact that the norm of the inclusion $\bar{A}^S_{\bar{\theta},q;J} \hookrightarrow \bar{A}^S_{\bar{\theta},q;K}$ only depends on $\bar{\theta}$, the result follows. ∎

Observe that inequality (10) follows easily from Theorem 3.2.

Our next result is a direct consequence of Theorem 3.1 and gives a necessary condition for J- and K-method to coincide.

COROLLARY 3.3. *Let $\Pi=\overline{P_1\cdots P_N}$ be a convex polygon, $(\alpha,\beta)\in \operatorname{Int}\Pi$, $1\leqslant q\leqslant\infty$ and let $\bar{A}=\{A_1, ..., A_N\}$ be a Banach N-tuple. If*

$$\bar{A}_{(\alpha,\beta),q;J}=\bar{A}_{(\alpha,\beta),q;K}$$

then for any barycentric coordinates $\bar{\theta}=(\theta_1, ..., \theta_N)$ of (α,β) with respect to the vertices $P_1, ..., P_N$, we have

$$\bar{A}^S_{\bar{\theta},q;J}=\bar{A}^S_{\bar{\theta},q;K}=\bar{A}_{(\alpha,\beta),q;J}=\bar{A}_{(\alpha,\beta),q;K}.$$

As a first application of Corollary 3.3 we show a simple N-tuple on which the J- and K-spaces do not coincide.

EXAMPLE 3.4. Let $\Pi=\overline{P_1\cdots P_N}$ be a convex polygon with at least 4 vertices ($N\geqslant 4$), let $(\alpha,\beta)\in\operatorname{Int}\Pi$ and $1\leqslant q\leqslant\infty$. Assume that (B_0, B_1) is a Banach couple such that $B_0\cap B_1$ is not closed in B_0+B_1. Set

$$A_j=\begin{cases} B_0 & \text{if } j=1, ..., N-1,\\ B_1 & \text{if } j=N.\end{cases}$$

Then the Banach N-tuple $\bar{A}=\{A_1, ..., A_N\}$ satisfies

$$\bar{A}_{(\alpha,\beta),q;J}\neq\bar{A}_{(\alpha,\beta),q;K}. \tag{13}$$

Indeed, one can choose barycentric coordinates $\bar{\theta} = (\theta_1, ..., \theta_N)$, $\bar{\eta} = (\eta_1, ..., \eta_N)$ of (α, β) with respect to $P_1, ..., P_N$ in such a way that $\theta_N \neq \eta_N$. By Corollary 3.3, in order to prove inequality (13), it suffices to show that

$$\bar{A}^S_{\bar{\theta}, q; K} \neq \bar{A}^S_{\bar{\eta}, q; K}. \tag{14}$$

Since $B_0 \cap B_1$ is not closed in $B_0 + B_1$, the classical real method on (B_0, B_1) depends effectively on its parameters (see [13, Thm. 3.1]), hence

$$(B_0, B_1)_{\theta_N, q} \neq (B_0, B_1)_{\eta_N, q} \qquad \text{because} \quad \theta_N \neq \eta_N.$$

Taking into account that the first $N-1$ spaces of the N-tuple $\bar{A}$ are all the same, it is not hard to check that

$$\bar{A}^S_{\bar{\theta}, q; K} = (B_0, B_1)_{\theta_N, q}$$

(see [16, Prop. 6.3]). Similarly

$$\bar{A}^S_{\bar{\eta}, q; K} = (B_0, B_1)_{\eta_N, q}.$$

This gives (14) and consequently (13).

Examples of Banach N-tuples where K- and J-methods coincide can be found in [7, Sect. 3]. Theorems 2.9 and 2.10 can also be used to construct Banach N-tuples having the coincidence property.

Note that Corollary 3.3 says, roughly speaking, that on N-tuples where K- and J-spaces coincide, the theory of methods associated to polygons is a special case of Sparr's theory. We close the paper with an illustration of this "principle." It refers to Fernandez' spaces, so Π is equal to the unit square.

EXAMPLE 3.5. One of the original motivations for Fernandez' work was to calculate the interpolation spaces generated by the 4-tuple of vector-valued Lebesgue spaces

$$\bar{X} = \{L_1(L_1), L_1(L_\infty), L_\infty(L_1), L_\infty(L_\infty)\}.$$

He stated [10] that if $0 < \alpha$, $\beta < 1$, $1/p = 1 - \alpha$ and $1/q = 1 - \beta$, then the resulting space is

$$L_q(L_{p,q})$$

but his proof has some inaccuracies (see [8, 11, 14]). More recently, he proved in [11, Thm. 4.9], that K- and J-spaces coincide on $\bar{X}$ without identifying the interpolation spaces.

Next, we give an alternative proof of the formula stated by Fernandez in [10].

Since $\bar{X}^F_{(\alpha,\beta),q;K} = \bar{X}^F_{(\alpha,\beta),q;J}$ and

$$(\alpha,\beta) = (1-\alpha)(1-\beta)(0,0) + \alpha(1-\beta)(1,0) + \beta(1-\alpha)(0,1) + \alpha\beta(1,1),$$

it follows from Corollary 3.3 that

$$\bar{X}^F_{(\alpha,\beta),q;K} = \bar{X}^S_{\bar{\theta},q;K},$$

where

$$\bar{\theta} = ((1-\alpha)(1-\beta), \alpha(1-\beta), \beta(1-\alpha), \alpha\beta).$$

Using now [16, Thm. 8.1], we have that

$$\begin{aligned} \bar{X}^S_{\bar{\theta},q;K} &= L_q((L_1, L_\infty, L_1, L_\infty)^S_{\bar{\theta},q;K}) \\ &= L_q((L_1, L_\infty)_{\alpha,q}) \\ &= L_q(L_{p,q}). \end{aligned}$$

ACKNOWLEDGMENT

We thank the referee for his comments on the proof of Theorem 1.9, which allowed us to simplify and shorten it.

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